Estimación de un modelo de equilibrio general dinámico y estocástico

- **El modelo DSGE:**

$$\underset{\{c_t, h_t, k_{t+1}\}}{Max} E_0 \sum_{t=0}^{\infty} \beta^t \left[\ln c_t - \gamma h_t\right] a_t$$

sujeto a: $\theta_t k_t^{\alpha} h_t^{1-\alpha} = c_t + k_{t+1} - (1-\delta) k_t$

k_0 dado

donde a_t y θ_t son dos shocks estructurales, el primero es un shock en preferencias (que llamaremos shock de demanda) y el otro es un shock en la productividad total de los factores (que llamaremos shock de oferta).

Condiciones del equilibrio

$$y_t = \theta_t k_t^{\alpha} h_t^{1-\alpha} \qquad (1)$$

$$\ln\theta_t = (1-\rho)\ln\overline{\theta} + \rho\ln\theta_{t-1} + \varepsilon_t, \qquad (2)$$

$$y_t = c_t + i_t, \qquad (3)$$

$$k_{t+1} = (1-\delta)k_t + i_t, \qquad (4)$$

$$\gamma c_t h_t = (1-\alpha)y_t, \qquad (5)$$

$$\frac{a_t}{c_t} = \beta E_t\left[\frac{a_{t+1}}{c_{t+1}}\left(\alpha\frac{y_{t+1}}{k_{t+1}} + 1 - \delta\right)\right], \qquad (6)$$

$$\ln a_t = \rho_a \ln a_{t-1} + \zeta_t, \qquad (7)$$

Estado estacionario

$$c_{ss} = \frac{1-\alpha}{\gamma}\bar{\theta}\left[\frac{\bar{\theta}\alpha}{(1/\beta)-1+\delta}\right]^{\frac{\alpha}{1-\alpha}},$$

$$k_{ss} = \frac{\alpha c_{ss}}{(1/\beta)-1+\delta-\alpha\delta},$$

$$h_{ss} = k_{ss}\left[\frac{(1/\beta)-1+\delta}{\bar{\theta}\alpha}\right]^{\frac{1}{1-\alpha}},$$

$$y_{ss} = \bar{\theta}k_{ss}^{\alpha}h_{ss}^{1-\alpha}$$

Log-linealización

Sea $\hat{x}_t = \ln(x_t / x_{ss}),\ \ x = y, c, k, i, h, a, \theta$

$$\hat{y}_t = \hat{\theta}_t + \alpha\hat{k}_t + (1-\alpha)\hat{h}_t, \tag{1}$$

$$\hat{\theta}_t = \rho\hat{\theta}_{t-1} + \varepsilon_t, \tag{2}$$

$$\left[\left(1/\beta\right) - 1 + \delta\right]\hat{y}_t = \left[\left(1/\beta\right) - 1 + \delta - \alpha\delta\right]\hat{c}_t + \alpha\delta\hat{i}_t, \tag{3}$$

$$\hat{k}_{t+1} = (1-\delta)\hat{k}_t + \delta\hat{i}_t, \tag{4}$$

$$\hat{c}_t + \hat{h}_t = \hat{y}_t, \tag{5}$$

$$\left(1/\beta\right)\hat{a}_t + \left(1/\beta\right)\hat{c}_t - \left(1/\beta\right)E_t\hat{c}_{t+1} + \left[\left(1/\beta\right) - 1 + \delta\right]\left(E_t\hat{y}_{t+1} - \hat{k}_{t+1}\right) - \left(1/\beta\right)E_t\hat{a}_{t+1}, \tag{6}$$

$$\hat{a}_t = \rho_a\hat{a}_{t-1} + \zeta_t \tag{7}$$

- **Transformando el modelo en una representación en espacio de los estados**

- *Ecuaciones contemporáneas*

$$\underbrace{\begin{bmatrix} 1 & 0 & \alpha-1 \\ \frac{1}{\beta}-1+\delta & -\alpha\delta & 0 \\ 1 & 0 & -1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} \hat{y}_t \\ \hat{i}_t \\ \hat{h}_t \end{bmatrix}}_{f_t^0}=\underbrace{\begin{bmatrix} \alpha & 0 \\ 0 & \frac{1}{\beta}-1+\delta-\alpha\delta \\ 0 & 1 \end{bmatrix}}_{B}\underbrace{\begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix}}_{s_t^0}+\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{C}\underbrace{\begin{bmatrix} \hat{\theta}_t \\ \hat{a}_t \end{bmatrix}}_{\vartheta_t}\Rightarrow$$

$$Af_t^0 = Bs_t^0 + C\vartheta_t \Rightarrow f_t^0 = A^{-1}Bs_t^0 + A^{-1}C\vartheta_t \ , \qquad (8)$$

Sea $P = \begin{bmatrix} \rho & 0 \\ 0 & \rho_a \end{bmatrix}$; entonces $E_t \vartheta_{t+1} = P \vartheta_t$

- ***Ecuaciones dinámicas (excepto los shocks estructurales)***

$$\underbrace{\begin{bmatrix} 1 & 0 \\ \frac{1}{\beta}-1+\delta & \frac{1}{\beta} \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \hat{k}_{t+1} \\ E_t\hat{c}_{t+1} \end{bmatrix}}_{E_t s_{t+1}^0} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{\beta}-1+\delta & 0 & 0 \end{bmatrix}}_{F} \underbrace{\begin{bmatrix} E_t\hat{y}_{t+1} \\ E_t\hat{i}_{t+1} \\ E_t\hat{h}_{t+1} \end{bmatrix}}_{E_t f_{t+1}^0} = \underbrace{\begin{bmatrix} 1-\delta & 0 \\ 0 & \frac{1}{\beta} \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix}}_{s_t^0} +$$

$$\underbrace{\begin{bmatrix} 0 & \delta & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} \hat{y}_t \\ \hat{i}_t \\ \hat{h}_t \end{bmatrix}}_{f_t^0} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\beta}(1-\rho_a) \end{bmatrix}}_{J} \underbrace{\begin{bmatrix} \hat{\theta}_t \\ \hat{a}_t \end{bmatrix}}_{\vartheta_t} \Rightarrow$$

$$DE_t s_{t+1}^0 + FE_t f_{t+1}^0 = Gs_t^0 + Hf_t^0 + J\vartheta_t \quad , \qquad (9)$$

De (8) y (9) tenemos que:

$$E_t s_{t+1}^0 = \mathrm{K} s_t^0 + L\vartheta_t , \qquad (10)$$

donde $\mathrm{K} \equiv \left(D + FA^{-1}B\right)^{-1}\left(G + HA^{-1}B\right)$

$$L \equiv \left(D + FA^{-1}B\right)^{-1}\left(J + HA^{-1}C - FA^{-1}CP\right)$$

Suponemos que K admite la descomposición de Jordan:

$$\mathrm{K} = M\Lambda M^{-1}; \quad \text{Sea } M^{-1} = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}, \Lambda = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}, |\mu_1| < 1, |\mu_2| > 1$$

El sistema de ecuaciones (10) puede ser expresado como sigue:

$$\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \begin{bmatrix} \hat{k}_{t+1} \\ E_t \hat{c}_{t+1} \end{bmatrix} = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix} \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix} + \underbrace{\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}}_{Q} \begin{bmatrix} \hat{\theta}_t \\ \hat{a}_t \end{bmatrix}$$

- ***Condición de estabilidad (resolviendo la segunda ecuación de*** **(Ω)** ***hacia adelante, aplicando la ley de expectativas iteradas)***

$$\hat{c}_t = -\frac{u_2}{v_2}\hat{k}_t + \frac{Q_{21}/v_2}{\rho - \mu_2}\hat{\theta}_t + \frac{Q_{22}/v_2}{\rho_a - \mu_2}\hat{a}_t$$

- ***Ecuación de estado (sustituyendo la condición de estabilidad en la primera ecuación de (Ω))***

$$\hat{k}_{t+1} = \mu_1 \hat{k}_t + \underbrace{\left[\frac{Q_{11} - \frac{\rho - \mu_1}{\rho - \mu_2} \frac{v_1}{v_2} Q_{21}}{u_1 - \frac{v_1}{v_2} u_2} \right]}_{N_1} \hat{\theta}_t + \underbrace{\left[\frac{Q_{12} - \frac{\rho_a - \mu_1}{\rho_a - \mu_2} \frac{v_1}{v_2} Q_{22}}{u_1 - \frac{v_1}{v_2} u_2} \right]}_{N_2} \hat{a}_t$$

- ***Representación en el espacio de los estados***

$$\begin{bmatrix} \hat{k}_{t+1} \\ \hat{\theta}_{t+1} \\ \hat{a}_{t+1} \end{bmatrix} = \begin{bmatrix} \mu_1 & N_1 & N_2 \\ 0 & \rho & 0 \\ 0 & 0 & \rho_a \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{\theta}_t \\ \hat{a}_t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{t+1} \\ \zeta_{t+1} \end{bmatrix},$$

$$\hat{c}_t = \begin{bmatrix} -\dfrac{u_2}{v_2} & \dfrac{Q_{21}/v_2}{\rho - \mu_2} & \dfrac{Q_{22}/v_2}{\rho_a - \mu_2} \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{\theta}_t \\ \hat{a}_t \end{bmatrix}$$

Además,

$$\left.\begin{array}{l}\hat{y}_t = \hat{\theta}_t + \alpha \hat{k}_t + (1-\alpha)\hat{h}_t \\[2ex] \hat{y}_t = \hat{c}_t + \hat{h}_t = -\dfrac{u_2}{v_2}\hat{k}_t + \dfrac{Q_{21}/v_2}{\rho-\mu_2}\hat{\theta}_t + \dfrac{Q_{22}/v_2}{\rho_a-\mu_2}\hat{a}_t + \hat{h}_t\end{array}\right\} \Rightarrow$$

$$\underbrace{\begin{bmatrix} 1 & \alpha-1 \\ 1 & -1 \end{bmatrix}}_{A1} \begin{bmatrix} \hat{y}_t \\ \hat{h}_t \end{bmatrix} = \underbrace{\begin{bmatrix} \alpha & 1 & 0 \\ -\dfrac{u_2}{v_2} & \dfrac{Q_{21}/v_2}{\rho-\mu_2} & \dfrac{Q_{22}/v_2}{\rho_a-\mu_2} \end{bmatrix}}_{B1} \begin{bmatrix} \hat{k}_t \\ \hat{\theta}_t \\ \hat{a}_t \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} \hat{y}_t \\ \hat{h}_t \end{bmatrix} = \underbrace{(A1)^{-1} B1}_{BT1} \begin{bmatrix} \hat{k}_t \\ \hat{\theta}_t \\ \hat{a}_t \end{bmatrix}$$

- **Filtro de Kalman**

Sistema general en espacio de los estados:

$$\underset{(r\times1)}{\xi_{t+1}} = \underset{(r\times r)}{F}\ \underset{(r\times1)}{\xi_t} + \underset{(r\times s)}{U}\ \underset{(s\times1)}{\upsilon_{t+1}}, \quad \text{ecuación de estado}$$

$$\underset{(n\times1)}{y_t} = \underset{(n\times k)}{A'}\ \underset{(k\times1)}{x_t} + \underset{(n\times r)}{H'}\ \underset{(r\times1)}{\xi_t} + \underset{(n\times1)}{w_t}, \quad \text{ecuación de control}$$

$$E\left(\upsilon_t\upsilon_\tau'\right) = \begin{cases} \underset{(s\times s)}{Q}, \text{ para } t = \tau \\ 0, \text{ para } t \neq \tau \end{cases}$$

$$E\left(w_t w_\tau'\right) = \begin{cases} \underset{(n\times n)}{R}, \text{ para } t = \tau \\ 0, \text{ para } t \neq \tau \end{cases}$$

Equivalencias con nuestro modelo teórico:

$$\xi_t \equiv \left[\hat{k}_t, \hat{\theta}_t, \hat{a}_t\right]', \upsilon_t \equiv \left[\varepsilon_t, \zeta_t\right], F \equiv \begin{bmatrix} \mu_1 & N_1 & N_2 \\ 0 & \rho & 0 \\ 0 & 0 & \rho_a \end{bmatrix}, U \equiv \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$y_t \equiv \left[\hat{c}_t, \hat{h}_t\right]' \text{ ó } y_t \equiv \left[\hat{y}_t, \hat{h}_t\right],$$

$$H' \equiv \begin{bmatrix} -\dfrac{u_2}{v_2} & \dfrac{Q_{21}/v_2}{\rho-\mu_2} & \dfrac{Q_{22}/v_2}{\rho_a-\mu_2} \\ BT_{21} & BT_{22} & BT_{23} \end{bmatrix}, \text{ ó } H' \equiv BT,$$

$$A' \equiv 0, x_t \equiv 0, w_t \equiv 0, \forall t, Q \equiv \begin{bmatrix} \sigma_\varepsilon^2 & 0 \\ 0 & \sigma_\zeta^2 \end{bmatrix}, R \equiv 0$$

Derivación del filtro:

Sean $\hat{\xi}_{t+1|t} \equiv \hat{E}\left(\xi_{t+1} \mid \Omega_t\right)$, $P_{t+1|t} \equiv \hat{E}\left[\left(\xi_{t+1} - \hat{\xi}_{t+1|t}\right)\left(\xi_{t+1} - \hat{\xi}_{t+1|t}\right)'\right]$,

donde $\Omega_t \equiv \left(y_t', y_{t-1}', \ldots, y_1', x_t', x_{t-1}', \ldots, x_1'\right)'$

Condiciones iniciales:

$\hat{\xi}_{1|0} = E(\xi_1) = 0$

Media incondicional: $E(\xi_{t+1}) = FE(\xi_t) \underset{E(\xi_{t+1})=E(\xi_t)}{\longrightarrow} (I_r - F)E(\xi_t) = 0 \underset{si\ \det(I_r - F)\neq 0}{\Longrightarrow}$

$$E(\xi_t) = 0$$

$$P_{1|0} = E\left\{\left[\xi_1 - E\left(\xi_1\right)\right]\left[\xi_1 - E\left(\xi_1\right)\right]'\right\} \rightarrow vec(P_{1|0}) = \left[I_{r^2} - (F \otimes F)^{-1}\right] vec(UQU').$$

Nótese que $\Sigma \equiv E\left(\xi_{t+1}\xi'_{t+1}\right) = E\left[\left(F\xi_t + U\upsilon_{t+1}\right)\left(F\xi_t + U\upsilon_{t+1}\right)'\right] = FE\left(\xi_t\xi'_t\right)F' + \ldots$

$$UE\left(\upsilon_{t+1}\upsilon'_{t+1}\right)U'$$

$$= F\Sigma F' + UQU' \Rightarrow$$

$$vec(\Sigma) = (F \otimes F)vec(\Sigma) + vec(UQU')$$

Calculando $\hat{\xi}_{t+1|t}$

$$K_t = FP_{t|t-1}H\left(H'P_{t|t-1}H + R\right)^{-1},$$

$$\hat{\xi}_{t+1|t} = F\hat{\xi}_{t|t-1} + K_t\left(y_t - A'x_t - H'\hat{\xi}_{t|t-1}\right),$$

$$P_{t+1|t} = FP_{t|t-1}F' - K_tH'P_{t|t-1}F' + UQU'$$

- **Estimación**

Si el estado inicial (ξ_1) y las innovaciones $\{w_t, \upsilon_t\}_{t=1}^{T}$ son gaussianas multivariantes, entonces $\hat{\xi}_{t+1|t}$ y $\hat{y}_{t+1|t}$ calculadas a través del filtro de Kalman, son óptimas y la distribución de y_t condicional sobre $\left(x_t, \Omega_{t-1}\right)$ es gaussiana:

$$y_t \,|\left(x_t, \Omega_{t-1}\right) \sim N\left(\left(A' x_t + H' \xi_{t|t-1}\right), \left(H' P_{t|t-1} H + R\right)\right)$$

Por tanto,

$$f_{Y_t|(X_t,\Omega_{t-1})} = (2\pi)^{-n/2} \left| H'P_{t|t-1}H + R \right|^{-1/2} \times$$
$$\exp\left\{ -\frac{1}{2}\left(y_t - A'x_t - H'\hat{\xi}_{t|t-1} \right)' \left(H'P_{t|t-1}H + R \right)^{-1} \left(y_t - A'x_t - H'\hat{\xi}_{t|t-1} \right) \right\}$$

Así, el logaritmo de la verosimilitud muestral es:

$$lfv = \sum_{t=1}^{T} \ln f_{Y_t|(X_t,\Omega_{t-1})}$$

El objetivo será encontrar los valores de los parámetros estructurales del modelo teórico que maximice el logaritmo de la verosimilitud muestral:

$\underset{\{\Theta\}}{Min} - lfv,$ donde Θ es el vector que contiene los parámetros estructurales.

- **Descomposición de la varianza**

Sea el modelo empírico:

$$s_{t+1} = As_t + B\varepsilon_{t+1}, \qquad (1)$$

$$d_t = Cs_t, \qquad (2)$$

$$V \equiv E(\varepsilon_{t+1}\varepsilon'_{t+1}), \text{ diagonal}, \qquad (3)$$

De (1): $(I-AL)s_{t+1}=B\varepsilon_{t+1}\Rightarrow s_t=\sum_{j=0}^{\infty}A^jB\varepsilon_{t-j}$

Así, $s_{t+k}=\sum_{j=0}^{\infty}A^jB\varepsilon_{t+k-j}$, $E_t\left(s_{t+k}\right)=\sum_{j=k}^{\infty}A^jB\varepsilon_{t+k-j}$, $\underbrace{s_{t+k}-E_t\left(s_{t+k}\right)}_{\substack{\text{error de previsión}\\ \text{a horizonte } k}}=\sum_{j=0}^{k-1}A^jB\varepsilon_{t+k-j}$

$$\underbrace{\Sigma_k^{(s)}}_{\substack{\text{varianza del error}\\ \text{de previsión}\\ \text{a horizonte } k}}=E\left[\left(s_{t+k}-E_t\left(s_{t+k}\right)\right)\left(s_{t+k}-E_t\left(s_{t+k}\right)\right)'\right]=$$

$$=BVB'+ABVB'A'+A^2BVB'\left(A^2\right)'+\ldots+A^{k-1}BVB'\left(A^{k-1}\right)'.$$

Además, $\lim_{k\to\infty}\Sigma_k^{(s)}=\Sigma^{(s)}$, donde $vec(\Sigma^{(s)})=\left[I_{size(s_t)}-A\otimes A\right]^{-1}vec(BVB')$

De (2): $$\Sigma_k^{(d)} = E_t\left[\left(d_{t+k} - E_t d_{t+k}\right)\left(d_{t+k} - E_t d_{t+k}\right)'\right]$$

$$= E_t\left[C\left(s_{t+k} - E_t s_{t+k}\right)\left(s_{t+k} - E_t s_{t+k}\right)' C'\right]$$

$$= C\Sigma_k^{(s)}C'$$

Además, $$\Sigma^{(d)} = \lim_{k\to\infty}\Sigma_k^{(d)} = C\Sigma^{(s)}C'$$